Published by Institute of Physics Publishing for SISSA/ISAS
Received: *January 11, 2004*
Accepted: *March 1, 2004*

Diagonal D-branes in product spaces and their Penrose limits

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Abstract: We study classes of D-branes embedded in various $\mathrm{AdS}^m \times S^n \times S^p \times T^q$ backgrounds, which nontrivially mix the target-space submanifolds. Mixing is achieved either via diagonal geometric embedding or through a mixed worldvolume flux which has one index in the sphere and one index in the AdS part. Branes of the former type wrap calibrated cycles in the target space, while those of the latter type wrap *non-supersymmetric* target space cycles which are stabilised only after the mixed worldvolume flux is turned on. In the second part of the paper we study two qualitatively different Penrose limits of these diagonal branes. In the first case we look at geodesics which *do not* belong to the worldvolume of brane. In order to get a nontrivial result, one needs to bring the brane closer and closer to the geodesic while taking the limit. The result is a D-brane with a worldvolume relativistic pulse. In the second case the Penrose geodesic belongs to the worldvolume and the resulting brane is of the "oblique" type: it is diagonally embedded between different SO groups of the target space pp-wave.

Keywords: D-branes, Penrose limit and pp-wave background.

JHEP03(2004)005

Contents

1. Introduction and summary

The study of D-branes in curved space hardly needs any further justification: from the pure CFT point-of-view they represent an interesting arena for the study of various CFT phenomena, they are crucial for the understanding of non-perturbative effects in the context of the AdS/CFT correspondence, and they are essential ingredients for the realisation of AdS/defect (conformal) field theory dualities.

In this paper we continue the study of D-branes in different curved manifolds. In the first part of the paper we address the issue of D-branes in manifolds which are products of several manifolds, while the branes themselves are *not* products of branes existing on these submanifolds. Instead, the branes nontrivially mix the target space submanifolds. Mixing

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is achieved via: 1) a *diagonal geometric embedding* or 2) through a *mixed worldvolume flux* which has one index in the sphere and one index in the AdS part, i.e. with a field strength of the schematic form $F_{\text{sphere,AdS}}$.

One way to construct the first class of branes is to start with supersymmetric flat space configurations of branes intersecting under angles. Replacing some of the branes by their supergravity solutions and subsequently focusing on their corresponding near-horizon geometries, while keeping the remaining branes as probes, one can derive the effective geometries for the probes. They nontrivially mix all submanifolds of the near horizon space. Because of their flat space origin we call them *diagonal* branes. Some of the diagonal branes have previously appeared in the literature. The diagonal $D3$ and $D5$ in $\text{AdS}_5 \times S^5$ have been constructed [1, 2], while various diagonal $M2$ and $M5$ branes in $\text{AdS}_4 \times S^7$ have been considered [3] in the context of generalised calibrations. In our study we however focus on different kinds of backgrounds.

In our study of D-branes in AdS spaces we mainly restrict to target spaces that are products of *group* manifolds: $\text{AdS}_3 \times S^3 \times T^4$ and $\text{AdS}_3 \times S^3 \times S^3 \times R$. In the first manifold, we consider a brane which mixes *all submanifolds* and interpolates between a brane with embedding $\text{AdS}_2 \times S^2 \times \text{point}$ and a brane with embedding $\text{AdS}_2 \times \text{point} \times T^2$. In the second target space we consider a brane which interpolates between a brane with embedding $\text{AdS}_2 \times S^2 \times \text{point} \times R$ and a brane with embedding $\text{AdS}_2 \times \text{point} \times S^2 \times R$. We derive the CFT symmmetries preserved by these branes, but due to the low number of preserved symmetries the construction of boundary states appears to be a hard problem. However, while a direct CFT analysis of these geometric diagonal branes does not look too tractable, an alternative and apparently more manageable approach is the one which uses the dCFT/AdS correspondence. Diagonal branes should correspond to RG-flows between *different* boundary conformal field theories. Especially interesting configurations are those of diagonal $D3$ and $D5$ branes in $\text{AdS}_5 \times S^5$, which deserve a separate study [4].

The first class of branes (which mixes submanifolds through a diagonal geometric embedding) can be described using calibration techniques. There are frequent statements in the literature that the flat space calibration results [5] (for example that supersymmetric branes wrap holomorphic cycles in the target space) can be extended in a straightforward manner to curved spaces. However, we point out that in cases when the full target space is not a complex manifold, the these results can be applied only to particular complex submanifolds for which we know in advance that it is consistent to truncate the DBI action to. A simple example is provided by an $\text{AdS}_3 \times S^3 \times T^4$ space in which we explicitly show that the calibration of the $D3$ brane can only be applied to submanifolds which contain a *maximal* S^2 in S^3.

The construction of the second type of diagonal branes is motivated by the *flat* space T-duality between branes under angles and branes with non-vanishing worldvolume fluxes. In curved spaces duality generically changes the background making these flat space relations less manifest. We show how in some cases these flat space dualities are inherited by the curved geometries. Starting with the flat space configuration of branes intersecting under an angle, we perform T-duality in such a way that the branes which will be replaced with the background *do not* carry any worldvolume flux, while a brane which will become a probe carries flux. Then we take a near horizon limit of this configuration. The near horizon brane wraps a *non-supersymmetric* target space cycle (a maximal $\mathrm{AdS}_4 \times S^4$ in $\mathrm{AdS}_5 \times S^5$) which is stabilised (becomes supersymmetric) only after the mixed worldvolume flux is turned on. This flux is different in nature from the one considered in [6, 7]: while the fluxes which are considered there modify the geometry of the brane, the flux which we consider *does not*. Maximal $\mathrm{AdS}_4 \times S^4$ is a solution to the DBI action with and without worldvolume flux and turning on the flux does not change the extrinsic curvature of the brane. This distinction is a consequence of the fact that the fluxes in [6, 7] originate from a brane ending on (and pulling) the other brane, while in our case it is a consequence of T-duality.

In the second part of this paper we consider various Penrose limits of the diagonal branes constructed in the first part. We use two types of geodesics. In the *first case* we look at a geodesic which *does not* belong to the worldvolume of the brane. In order to get a nontrivial result, one needs to bring the brane closer and closer to the geodesic while taking the limit. The resulting D-brane is a brane with a relativistic pulse propagating on its worldvolume. These branes have recently been analysed in *flat* space in [8]. In the *second case* the Penrose geodesic belongs to the worldvolume of the diagonal brane and the resulting brane is of the "oblique" type: it is diagonally embedded between different SO groups of the target space pp-wave. These kinds of branes have been recently discovered in [9], and further analysed in [10]. Here we point out their AdS origin.

At the end of the paper we also analyse the Penrose limit of branes with diagonal flux. Generically, in order to obtain a finite flux in the Penrose limit for these branes, one needs to simultaneously scale the geodesic to be closer and closer to the brane surface, and require it to become *orthogonal* to the worldvolume flux. The Penrose limit of a brane with diagonal flux leads to a D-brane with null flux.

2. Diagonal branes from branes under the angle in flat space

The first class of D-branes we want to study are D-branes which mix various target space submanifolds in a *geometric* way. As explained in the introduction these can be derived from the (supersymmetric) flat space configurations of branes intersecting under angles. More precisely we consider the following supersymmetric flat space configurations:

$$\begin{array}{lcccccccccc} F1 & 0 & 1 & - & - & - & - & - & - & - & - \\ NS5 & 0 & 1 & 2 & 3 & 4 & 5 & - & - & - & - \\ D3' & 0 & - & - & - & - & - & 6 & 7 & 8 & - \\ D3'' & 0 & - & 2 & 3 & - & - & 6 & - & - & - \end{array} \tag{2.1}$$

and

$$\begin{array}{lcccccccccc} F1 & 0 & 1 & - & - & - & - & - & - & - & - \\ NS5 & 0 & 1 & 2 & 3 & 4 & 5 & - & - & - & - \\ NS5 & 0 & 1 & - & - & - & - & 6 & 7 & 8 & 9 \\ D4' & 0 & - & 2 & 3 & 4 & - & 6 & - & - & - \\ D4'' & 0 & & - & - & 4 & - & 6 & 7 & 8 & - \end{array} \tag{2.2}$$

The second configuration can be derived from the eleven-dimensional configuration of a "non-standard" brane intersection by dimensional reduction in direction 10 [11]:

$$
\begin{array}{lccccccccccc}
M2 & 0 & 1 & - & - & - & - & - & - & - & - & 10 \\
M5 & 0 & 1 & 2 & 3 & 4 & 5 & - & - & - & - & - \\
M5 & 0 & 1 & - & - & - & - & 6 & 7 & 8 & 9 & - \\
M5' & 0 & - & 2 & 3 & 4 & - & 6 & - & - & - & 10 \\
M5'' & 0 & & - & - & 4 & - & 6 & 7 & 8 & - & 10
\end{array}
\tag{2.3}
$$

The unprimed branes are subsequently replaced with their corresponding supergravity solutions and finally with the near horizon geometries, while the primed branes are treated as probes. A generic probe brane is taken to be an arbitrary brane which interpolates between the two reference (primed) branes marked in the table. We will see that knowing the flat space embedding of the probe branes suggests the ansatz which solves the effective equations of motion in the near horizon limit.

Note also that due to the way we construct branes in the near horizon geometries (namely starting from physically acceptable brane configurations in flat space), we are guaranteed that the final brane configurations are "good" branes, i.e. that they give rise to consistent CFTs.

2.1 Diagonal $D3$ brane in the $\mathrm{AdS}_3 \times S^3 \times T^4$ background

The near horizon geometry of the intersecting branes in (2.1) (excluding the primed, probe branes) is $\mathrm{AdS}_3 \times S^3 \times T^4$. To keep the connection to the flat space picture manifest we choose to write the metric and flux in the $\mathrm{AdS}_3 \times S^3$ space in cartesian and Poincaré coordinates, respectively given by[1]

$$
\begin{aligned}
ds^2 &= R^2u^2\big(-dt^2+dx_1^2\big)+R_t^2\left(dx_2^2+\cdots+dx_5^2\right)+\frac{1}{R^2u^2}\big(dx_6^2+\cdots+dx_9^2\big) && (2.5)\\
&= R^2u^2\big(-dt^2+dx_1^2\big)+R_t^2\left(dl^2+l^2d\mu^2+dx_4^2+dx_5^2\right)+\frac{R^2du^2}{u^2}+\\
&\quad +R^2\Big(d\psi^2+\sin^2\psi\left(d\xi^2+\sin^2\xi d\eta^2\right)\Big) && (2.6)\\
H &= -2R^2u\,dt\wedge dx_1\wedge du+2R^2\sin^2\psi\,\sin\xi\,d\psi\wedge d\xi\wedge d\eta\,, && (2.7)
\end{aligned}
$$

where $u^2=(x_6^2+x_7^2+x_8^2+x_9^2)/R^4$. The dimensionless radius of $\mathrm{AdS}_3\times S^3$ is denoted by R and R_t is the dimensionless radius of the T^4. These two parameters are independent. In the remainder of this paper we will always choose to work, for all metrics, with dimensionless coordinates (so that ds^2 and H are also dimensionless). This will be the reason that later, when we take the Penrose limit of various metrics, we will only scale dimensionless parameters, while keeping α' fixed.

---

[1] In order to go from cartesian (2.5) to Poincaré coordinates (2.6) one uses the transformations

$$
\begin{aligned}
&x_9 = R^2u\cos\psi\,, \quad x_6 = R^2u\sin\psi\cos\xi\,, \quad x_7 = R^2u\sin\psi\sin\xi\cos\eta\,, \quad x_8 = R^2u\sin\psi\sin\xi\sin\eta\,,\\
&x_2 = l\cos\mu\,, \quad x_3 = l\sin\mu\,.
\end{aligned}
\tag{2.4}
$$

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2.1.1 The DBI analysis and remarks about calibrations in curved spaces

Let us consider a $D3$ brane which intersects the $F1 - NS5$ system in flat space as in (2.1) and is placed under the angle α in the $2-7$ and $3-8$ planes. In cartesian coordinates its embedding is given by the following equations

$$x_1 = x_4 = x_5 = x_9 = 0\,, \qquad x_2 = \tan\alpha\, x_7\,, \qquad x_3 = \tan\,\alpha x_8\,. \tag{2.8}$$

Equivalently, if we introduce complex coordinates $\omega = x_7 + ix_8$ and $z = x_2 + ix_3$ these equations become

$$f(z,\omega) = z - \tan\alpha\,\omega = 0\,. \tag{2.9}$$

More generally it is known from Kähler calibrations in *flat* space [12, 13, 5] that all complex submanifolds of $\mathbb{C}^n$ are volume minimising, and hence solve the D-brane action in the absence of worldvolume fluxes. In other words, any holomorphic function $f(z,\omega) = 0$ is a solution to the DBI action in *flat* space. Different functions have a target space interpretation of a system of two, three etc. Dp/M branes which intersect over a $D_{(p-2)}/M_{(p-2)}$ brane, and are rotated by arbitrary SU(2) angles.[2]

We expect that the same result holds in the curved background, namely if we replace the $F1 - NS5$ system with (2.5) and treat the $D3$ brane as a probe in this geometry. To prove this, let us use the static gauge in which we identify the worldvolume coordinates σ^i with a subset of target space coordinates as $\sigma^i = (t, x_6, \omega, \bar\omega)$ and treat x_i $(i = 1,4,5,9)$, z and $\bar z$ as transverse scalars. We want to check that

$$\mathcal{F}_{ij} = 0\,, \qquad z = f(\omega,\bar\omega)\,, \qquad x_9 = C = \text{const.}\,, \qquad x_i = 0 \quad (i = 1,4,5)\,, \tag{2.10}$$

is a solution to the full DBI equations of motion given in (A.21) and (A.22), for any holomorphic function f. Note that we have also put the transverse scalar x_9 to be an *arbitrary* constant since we want to show that, unless $C = 0$, the above configuration is not a solution to the near-horizon equations of motion.

Let us first consider the second equation in (A.22), obtained by varying the DBI action with respect to the worldvolume scalars. For an arbitrary holomorphic function f and $x_9 = C = $ const., the z (and $\bar z$) components of this equation reduce to $\partial_{\bar\omega}\partial_\omega f \equiv 0$. The x_9 component reduces to

$$\frac{4x_9}{u^4R^2} = 0\,, \qquad u^2 = x_6^2 + |\omega|^2 + x_9^2\,. \tag{2.11}$$

We see that $x_9 = $ const. $\neq 0$ is *not* a solution to the equations of motion in the near horizon limit. The reason for this can be seen in flat space: when separating the $D3$ from the $NS5 - F1$ system there is an anomalous creation of an $F1$ string [14] which pulls the $D3$ brane toward the $D1 - D5$ system and acts as a source for the worldvolume flux. Hence in order to obtain the near horizon solution describing a finite x_9 separation one has to turn on the electric field on the brane worldvolume, as in [6].

Note also that although for our configuration the separation in the direction x_9 is not a free modulus, this is not always the case. For example for the $D3 - D7$ system intersecting

[2] For the SU(2) rotations, all branes are rotated in the same two two-planes.

over a $D3$ brane $(D3, D7|3+1)$, or a $(D3, D5|2+1)$ system, the separation (Higgsing) of branes in any common transverse direction can be done for "free": the solution $x^{\text{transverse}} = C = \text{const.}$ is a supersymmetric solution in flat and AdS $\times$ S spaces [15]. However, note that when taking the near horizon limit in order to obtain the nontrivial configuration, one sends not only u to zero but one has to *simultaneously* scale the constant C to zero. This is similar to the situation we will encounter when taking various Penrose limits in section 4.1.

We also have to check that the equation (A.21), obtained by varying the DBI action with respect to the gauge potential, vanishes. Since there are no RR fields in the target space, the right-hand side of (A.21) is identically zero. The rest is solved by (2.10) with $C = 0$, since the antisymmetric part of the inverse matrix M^{ij} is [16]

$$\theta^{ij} = \left(\frac{1}{g_{\text{induced}} + \mathcal{F}} \mathcal{F} \frac{1}{g_{\text{induced}} - \mathcal{F}} \right)^{ij} \tag{2.12}$$

i.e. it vanishes whenever $\mathcal{F} = 0$.

The brane configuration given above can be also described using calibrations. Following the discussion of calibrated surfaces in curved spaces [2, 3] we can write the calibration two-form (calibrating the two-dimensional surface in the $z, \bar{z}, \omega, \bar{\omega}$ subspace) as

$$\varphi = dz \wedge d\bar{z} + \frac{1}{R^2 \omega \bar{\omega}} d\omega \wedge d\bar{\omega} \,. \tag{2.13}$$

We would like to point out two important points regarding the construction of solutions to the DBI action in curved spaces using calibrations. Firstly, by construction, the calibration method produces minimal surfaces.[3] Since the DBI action is not only a geometrical action, establishing the existence of an extremal point in the truncated part of the full configuration space does not a-priori imply the existence of an extremal or saddle point in the full configuration space. Hence, one has to check separately whether the minimal surface thus constructed is compatible with a truncation of the full DBI action to its geometric part (i.e. the Dirac action).

Secondly, the calibration method generically produces minimal surfaces in *submanifolds* of the full target space. This is basically dictated by the fact that one needs globally well defined (almost) complex structures, which generically can only be defined only on submanifolds (of even dimension). However, the choice of the submanifold on which we construct the calibration form is not a-priori restricted. For example, had we chosen the $x_1 = x_4 = x_5 = 0\,,\, x_9 = \text{const.} \neq 0$ submanifold (as opposed to the submanifold for which this constant is zero) to construct the calibration two-form (2.13), we would have ended up with a minimal two-surface on this submanifold. This one does not, however, lift to an extremal two-surface in the full space, as one can see from equation (2.11).[4] Hence, again, one needs to make sure that the truncation which one makes when restricting to a submanifold is a consistent one.

[3]Here, we are excluding generalisations of the type presented in [2, 3].

[4]Note that the $x_9 = 0$ surface actually lifts to a *maximal* two sphere in the full space. This is just an illustration of the fact that a minimum in a subspace of the full configuration space can lift to a saddle point in the full configuration space.

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2.1.2 The geometry of the diagonal brane

Let us now try to understand the geometry of a *generic* diagonal $D3$ brane. In global coordinates (for definitions of various coordinate systems see (2.4) as well as (A.15) and (A.20) in the appendix) a generic diagonal $D3$ is given by the equations

$$\begin{aligned} \psi &= \frac{\pi}{2}\,, \qquad \sigma = 0\,, \qquad x_4 = x_5 = 0\,, \\ f(z,\omega) &= 0\,, \qquad \omega = R^2 \sin\xi \big(\cosh\rho \cos\tau - \sinh\rho \big)\, e^{i\eta}\;. \end{aligned} \tag{2.14}$$

As before f is a holomorphic function of z and ω. To understand the geometry of the brane, let us first understand the simple example of the holomorphic function

$$f(z,\omega) = (z - \tan\alpha\,\omega)\omega - c\,. \tag{2.15}$$

We see that there are two asymptotic solutions to the equation $f(z,\omega) = 0$,

$$|\omega| \ll |c| \qquad |z| \to \infty\,, \tag{2.16}$$

$$|\omega| \gg |c| \qquad z \to \tan\alpha\,\omega\,. \tag{2.17}$$

These correspond to two different $D3$ branes: the first one is embedded along the $T^2(z)$ torus in T^4 and the second one is diagonally embedded between the $S^2 \times \rho$ part in $\mathrm{AdS}_3 \times S^3$ and the $T^2(z)$ torus in T^4. So, similar to the situation in flat space, the function (2.15) describes two $D3$ branes that intersect under the angle $\frac{\pi}{2} - \alpha$ in the $\mathrm{AdS}_3 \times S^3 \times T^4$ space.

To see this in more detail, let $\alpha = 0$ (this is the case of two orthogonally intersecting branes) and focus on the region near the boundary, $\rho \to \infty$. Then we have

$$|\omega| = \frac{1}{2} R^2\, e^{\rho}\, \big| \sin\xi\, (\cos\tau - 1) \big|\,. \tag{2.18}$$

We see that for $\xi \neq 0$ and $\tau \neq 0 \pmod{2\pi}$, $|\omega|$ is large, so we recover the asymptotic solution (2.17) which is the $\mathrm{AdS}_2 \times S^2$ brane. As we approach the origin of AdS_3 ($\rho \to 0$), $|\omega|$ becomes

$$|\omega| = R^2 \big| \sin\xi (\cos\tau - \rho) \big| \leq R^2\,. \tag{2.19}$$

So by choosing the constant c in (2.15) larger than R^2 we see that (2.16) holds, hence the brane asymptotes to the $\mathrm{AdS}_2 \times T^2(z)$ brane. In summary, as we move in the radial direction toward the center of AdS the brane system interpolates between the $\mathrm{AdS}_2 \times S^2$ brane and the $\mathrm{AdS}_2 \times T^2$ brane. If $\alpha \neq 0$ then the asymptotic region of large ρ corresponds to a *single* non-conformal brane (2.9), which nontrivially mixes the AdS, sphere and torus parts, while the $\rho \to 0$ region remains unchanged.

These kind of interpolating brane solutions are especially interesting in the light of the defect (conformal) field theory/AdS correspondence. Particularly interesting examples are those of $D5$ and $D7$ branes in $\mathrm{AdS}_5 \times S^5$ which deserve a separate study [4]. These embeddings of branes are supposed to correspond to an RG flow between two different defect conformal field theories.

Finally, let us conclude this section with a brief comment on generalisations of the function f in (2.10) to higher degree (holomorphic) polynomials. These generalisations lead to configurations of multiply-intersecting brane systems, all rotated in the same two two-planes $2-7$ and $3-8$ but with different angles. In order to describe SU(3) and SU(4) rotations (in three and four two-planes) one needs to excite four (z_1, z_2) and six (z_1, z_2, z_3) transverse scalars respectively. The results are the similar to those that we have presented for the SU(2) case.

2.1.3 CFT symmetries of the diagonal branes

In the context of CFT constructions of D-branes one usually starts from a particular set of conditions imposed on the CFT currents, and then one deduces the effective geometry associated to these D-branes. Here, we are facing an opposite problem. The D-branes which were constructed in the previous sections are given by their effective, geometric embeddings. We would now like to deduce something about the CFT symmetries that they preserve.

Our strategy will be the following. We will first determine the *geometric* isometries preserved by the effective D-brane hypersurface. We will then try to argue how these geometrical isometries get lifted to worldsheet symmetries.[5] This approach, however, cannot directly tell us about the conserved charges that are stringy in nature (winding charges, for example).

We will restrict our analysis to the simplest case of the diagonal $D3$ brane (2.8). In the AdS-Poincaré coordinates and cylindrical coordinates in the $2-3$ plane, as given in (2.6), this brane is described by an effective embedding equation

$$x_2^2 + x_3^2 = \alpha^2 u^2 \sin^2\xi\,, \qquad \frac{x_2}{x_3} = \tan\eta\,, \qquad \psi = \frac{\pi}{2}\,, \qquad x_1 = x_4 = x_5 = 0\,. \tag{2.20}$$

We see that this surface nontrivially mixes the AdS$_2$, S^2 and T^2 submanifolds of the AdS$_3\times S^3\times T^4$ background. As a first step in the analysis we need to determine the isometries preserved by the AdS$_2$ submanifold in AdS$_3$ and the S^2 submanifold in S^3.

An AdS$_2$ submanifold in AdS$_3$ is defined by a twined conjugacy class

$$\mathcal{C}_{(\omega,g)} = \Big\{\omega(h) g h^{-1} |\ \forall h\in \mathrm{SL}(2,\mathbb{R})\ \Big\} \quad \text{with } \omega(h) = \omega_0^{-1} h\omega_0\,, \quad \omega_0 = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}. \tag{2.21}$$

Here $\omega(h)$ is an outer automorphism of the group. Note that all elements in the same conjugacy class have a fixed value of the trace, equal to $\mathrm{tr}(\omega_0 g)$. The AdS$_3$ metric is invariant under *separate* left and right group multiplications. These are related to two copies of the SL$(2,\mathbb{R})$ currents in the bulk CFT. The submanifold (2.21) (i.e. the constant trace) is invariant only under the *simultaneous* right and twisted left multiplications

$$\mathrm{tr}\left(\omega_0 g\right) = \mathrm{tr}\left(\omega_0(\omega_0^{-1} h\omega_0) g h^{-1}\right). \tag{2.22}$$

These isometries are related to the following set of worldsheet currents (on an open string which is restricted to move on the AdS$_2$ surface)

$$J = \omega_0 \bar{J}\omega_0\,. \tag{2.23}$$

[5]The geometrical isometries lead to conserved currents on the worldline of a point particle. In that sense the effective geometry is the geometry as seen by a point particle.

However, our brane is a subsurface in AdS_2, hence it breaks the symmetries further. To determine the residual isometry we will need the transformation properties of the AdS_2 coordinates under (2.22). This can easily be determined using the parametrisation (A.16) and (A.14). The coordinate u transforms as

$$u' = D^2 u - 2\frac{C\,D}{R} u\,t - \left(\frac{C}{R}\right)^2 \left(\frac{1}{u} + u(-t^2 + x_1^2)\right), \tag{2.24}$$

where A, B, C and D are parameters of an arbitrary $\mathrm{SL}(2,\mathbb{R})$ matrix. Note that for our brane (2.8) the AdS coordinate x_1 is zero. We see that under the action of the matrix

$$h = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} = e^{\frac{B}{2}(\sigma^+)}, \qquad \sigma^+ = \sigma^1 + i\sigma^2, \tag{2.25}$$

the coordinate u does not change.

For the isometries involving a sphere part of the configuration, note that an arbitrary two sphere S^2 in an SU(2) manifold is given by the conjugacy class

$$\mathcal{C}_g = \left\{ hgh^{-1} | \, \forall h \in \mathrm{SU}(2) \right\}. \tag{2.26}$$

It is invariant under the adjoint action $g \to h^{-1}gh$. This implies that the SU(2) currents $J_{\mathrm{SU}(2)}, \bar{J}_{\mathrm{SU}(2)}$ preserved by the S^2 brane are

$$J_{\mathrm{SU}(2)} = \bar{J}_{\mathrm{SU}(2)}\,. \tag{2.27}$$

For the isometries of our brane we will also need the transformation properties of the coordinate ξ under the adjoint action $(g \to h^{-1}gh)$,[6]

$$\cos\xi' = (aa^* - bb^*)\cos\xi - \mathrm{Re}(ab^*)\sin\xi\cos\eta - \mathrm{Im}(ab^*)\sin\xi\sin\eta\,. \tag{2.28}$$

Here a, b are the complex parameters of an SU(2) matrix.[7] We see that ξ is unchanged if and only if $b = 0$. This transformation is just a $\mathrm{U}(1)_\eta$ symmetry: it acts on η as $\eta \to \eta + \alpha$ (where $\alpha = \text{const.}$ is a parameter of the transformation) while keeping ξ and ψ unchanged.

Next, we need to determine which of the above symmetry transformations leave the conditions (2.20) unchanged. The first condition is invariant only under the transformations which separately preserve the u, ξ, x_2 and x_3 coordinates: these are the $\mathrm{U}(1)_{\sigma^+}$ and $\mathrm{U}(1)_\eta$ rotations. The second condition however breaks the $\mathrm{U}(1)_\eta$ isometry, implying that the only preserved current which originates from the AdS_3 and S^3 isometries is

$$J_+ = \omega_0 \bar{J}_+ \omega_0\,. \tag{2.30}$$

[6]Since our brane mixes only the S^2 given by $\psi = \pi/2$ with the AdS_2 factor, we do not need a generic transformation, but we restrict here to the big S^2 in S^3.

[7]We write a generic SU(2) matrix as

$$\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \quad |a|^2 + |b|^2 = 1\,. \tag{2.29}$$

As far as the isometries of the torus are concerned, the conditions (2.20) obviously break the $\mathrm{U}(1)_{x_i}$ isometries (with $i = 2,3,4,5$). However, since the directions x_4 and x_5 are Dirichlet, we know that the (stringy) winding currents

$$J_T^4 = -\bar{J}_T^4\,, \qquad J_T^5 = -\bar{J}_T^5\,, \tag{2.31}$$

are preserved. On the other hand, the conditions on x_2 and x_3 are neither Dirichlet nor Neumann conditions, and to determine the (stringy) currents associated to the first two conditions in (2.20), one needs to analyse the full sigma model subject to these boundary conditions. We will not perform this analysis here.

2.2 Diagonal $D4$ brane in the $\mathrm{AdS}_3 \times S^3 \times S^3 \times R$ background and its symmetries

Another interesting configuration of a diagonal D-brane can be obtained from the configuration (2.2) of intersecting branes. As before, we replace the first three D-branes in (2.2) by the near-horizon limit of the corresponding supergravity solution and treat the $D4$ brane as a probe. The probe can interpolate between the two reference positions marked in (2.2).

The metric of the near-horizon limit of the supergravity solution for the bound state of the $N_F^{(1)}$ $NS5_1$ branes, the $N_F^{(2)}$ $NS5_2$ branes and the N_T fundamental strings is [17]

$$ds^2 = \frac{r^2 r'^2}{N_T}\Big(-dt^2 + dx_1^2\Big) + \frac{N_F^{(2)}}{r^2}\Big(dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2\Big) + \frac{N_F^{(1)}}{r'^2}\Big(dx_6^2 + dx_7^2 + dx_8^2 + dx_9^2\Big)\,, \tag{2.32}$$

where $r^2 = x_2^2 + x_3^2 + x_4^2 + x_5^2$ and $r'^2 = x_6^2 + x_7^2 + x_8^2 + x_9^2$. As before, we have factored out the dimensionful parameter α' so that all coordinates and all parameters are dimensionless. It is convenient to introduce new variables [18, 19]

$$u = \frac{rr'}{N_T^{1/2} R}\,, \qquad \lambda = R\left(\sqrt{\frac{N_F^{(1)}}{N_F^{(2)}}}\log r - \sqrt{\frac{N_F^{(2)}}{N_F^{(1)}}}\log r'\right)\,, \qquad \frac{1}{R^2} = \frac{1}{N_F^{(1)}} + \frac{1}{N_F^{(2)}}\,. \tag{2.33}$$

In these variables the metric (2.32) and the accompanying NS 3-form can be written as

$$ds^2 = R^2\left(u^2(-dt^2 + dx_1^2) + \frac{du^2}{u^2}\right) + d\lambda^2 + N_F^{(2)} d\Omega_{(1)}^2 + N_F^{(1)} d\Omega_{(2)}^2 \tag{2.34}$$

$$H = 2\,R^{-1}\epsilon_{\mathrm{AdS}_3} + 2\,N_F^{(2)}\epsilon_{S^{3(1)}} + 2\,N_F^{(1)}\epsilon_{S^{3(2)}}\,. \tag{2.35}$$

Here $d\Omega_{(1)}^2$ and $\epsilon_{S^{3(1)}}$ are the metric and volume forms for the unit three-sphere in the space spanned by the x_2, x_3, x_4, x_5 coordinates. Similarly, $d\Omega_{(2)}^2$ and $\epsilon_{S^{3(2)}}$ are the metric and volume form for the unit three-sphere in the space spanned by the x_6, x_7, x_8, x_9 coordinates. We see that (2.34) and (2.35) manifestly exhibit the $\mathrm{AdS}_3 \times S^3 \times S^3 \times S^1$ structure of the near-horizon geometry.

To find the effective geometry for the diagonal $D4$ probe brane in this background we follow the same logic as in the previous section. As before, it can easily be shown that the embedding given by

$$x_1 = x_5 = x_9 = 0\,, \qquad x_7 + ix_8 = f(x_2 + ix_3)\,, \qquad \mathcal{F} = B + F = 0\,, \tag{2.36}$$

with f an arbitrary holomorphic function, solves the DBI action. This is in agreement with the general statements about calibrated surfaces in curved spacetime. The calibration two-form (calibrating the x_2, x_3, x_7, x_8 part of the D-brane surface) is

$$\varphi = \frac{1}{x_2^2 + x_3^2} dx_2 \wedge dx_3 + \frac{1}{x_7^2 + x_8^2} dx_7 \wedge dx_8 \,. \tag{2.37}$$

To say something about the symmetries and currents preserved by this type of brane we restrict to the simpler case of a brane with embedding

$$x_1 = x_5 = x_9 = 0 \,, \qquad x_2 = \tan \alpha x_7 \,, \qquad x_3 = \tan \alpha x_8 \,. \tag{2.38}$$

Using the coordinates (A.20) and (2.33) this surface can be written as

$$\begin{aligned} x_1 &= 0 \,, \qquad \psi_1 = \psi_2 = \frac{\pi}{2} \,, \qquad \eta_1 = \eta_2 \,, \\ \left(\frac{r}{r'}\right)^{a+\frac{1}{a}} &= \left(\tan \alpha \frac{\sin \xi_2}{\sin \xi_1}\right)^{a+\frac{1}{a}} = (R\sqrt{N_T})^{\frac{1}{a}-a} e^{\frac{2\lambda}{R}} u^{\frac{1}{a}-a} \,, \end{aligned} \tag{2.39}$$

where $a = \sqrt{N_F^{(1)}/N_F^{(2)}}$. We see that for generic values of the parameters, the $D4$ brane mixes in a nontrivial way all four submanifolds. However, when $a = 1$ the u-dependence in (2.39) drops and one is left with a brane that has the geometry of an AdS_2 brane times a 3-brane in the $S^3 \times S^3 \times S^1$ subspace. This brane is defined by the equations

$$\psi_1 = \psi_2 = \frac{\pi}{2} \,, \qquad \eta_1 = \eta_2 \,, \qquad e^{\frac{\lambda}{R}} = \tan \alpha \frac{\sin \xi_2}{\sin \xi_1} \,. \tag{2.40}$$

It is easy to find the symmetries preserved by this $D4$ brane. It trivially inherits those of the AdS_2 brane. For the three-dimensional part wrapping the spheres, we note that the first and third condition in (2.40) are preserved by separate U(1) rotations in the directions η_1 and η_2. Preservation of the second condition, however, relates these currents as

$$K_3 + \bar{K}_3 = L_3 + \bar{L}_3 \,, \tag{2.41}$$

where K_3,$\bar{K}_3$ and L_3,$\bar{L}_3$ are the left/right currents corresponding to $\mathrm{U}(1)_{\eta_1}$ and $\mathrm{U}(1)_{\eta_2}$ symmetries.

Another interesting brane can be derived by taking the holomorphic function f in (2.36) to be $z = \beta/w$. Using the coordinates (A.20) and the definition of u in (2.33) we get a $D4$ brane with the embedding[8]

$$x_1 = 0 \,, \qquad \psi_1 = \psi_2 = \frac{\pi}{2} \,, \qquad \eta_1 = -\eta_2 \,, \qquad u = \frac{\beta}{R\sqrt{N_T}} \frac{1}{\sin \xi_1 \sin \xi_2} \,. \tag{2.42}$$

Now the geometry of the D4-brane is a product of a D3-brane on $\mathrm{AdS}_3 \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ defined by (2.42) and a U(1) circle wrapped by λ.

[8]These formulae are valid only when $\beta \neq 0$.

3. Turning on a non-diagonal worldvolume flux

Up to now, our analysis has been restricted to situations where $\mathcal{F} = 0$. The mixing between the subspaces of the target space manifold was geometric: the brane worldvolumes were embedded in the target space in a diagonal way between the two submanifolds. We will now show that there is yet another way of implementing a mixing between target space submanifolds. Namely, for branes whose worldvolume is a product AdS $\times$ S it is possible to turn on a worldvolume flux which mixes the AdS and S directions. Moreover, unlike in the other known cases of branes with fluxes in AdS $\times$ S spaces [6], the presence of the non-diagonal flux in these cases does not modify the brane geometry.[9]

In a flat space it is well known that performing a T-duality transformation under an angle leads to brane configurations with non-vanishing fluxes. Hence using this property it is possible to "trade" non-orthogonally intersecting brane systems for orthogonally intersecting ones with flux. We want to show that a similar logic goes through in a curved target space.

Our starting point is the supersymmetric flat space configuration,

$$\begin{array}{llllllllll} D3 & 0 & - & 2 & 3 & - & - & - & - & 8 & - \\ D5' & 0 & - & 2 & - & 4 & 5 & 6 & - & 8 & - \\ D5'' & 0 & 1 & - & - & 4 & 5 & 6 & 7 & - & - \end{array} \tag{3.1}$$

We focus on the $D5$ brane which in a supersymmetric way intersects the "reference" $D5'$ and $D5''$ branes under an angle α,

$$x_2 = \tan\alpha\, x_1\,, \qquad x_8 = \tan\alpha\, x_7\,. \tag{3.2}$$

It extends in directions $4, 5$ and 6. Applying a T-duality transformation in the directions 1 and 8 leads to the following configuration:

$$\begin{array}{lllllllllll} D3 & 0 & 1 & 2 & 3 & - & - & - & - & - & - \\ D5' & 0 & 1 & 2 & - & 4 & 5 & 6 & - & - & - \\ D5'' & 0 & - & - & - & 4 & 5 & 6 & 7 & 8 & - \\ D7 & 0 & 1 & 2 & - & 4 & 5 & 6 & 7 & 8 & - \end{array} \tag{3.3}$$

The $D7$ brane originates from the diagonal $D5$ brane. Note that the $D7$ carries nontrivial worldvolume flux

$$F_{12} = -\cot\alpha\,, \qquad F_{78} = \tan\alpha\,, \tag{3.4}$$

due to the initial diagonal embedding. Note also that, in the absence of fluxes, the configuration (3.3) is *non-supersymmetric* and hence no-longer T-dual to (3.1).

[9] In [6] a configuration of an $\mathrm{AdS}_2 \times S^2$ brane with electric and magnetic worldvolume fluxes has been considered. It was shown that one can turn on electric/magnetic fields which are *each separately* proportional to the volume forms of the AdS_2/S^2 parts, and still preserve supersymmetry. The effect of the magnetic flux on the geometry of the embedded brane was that it changed (decreased) the size of the S^2 sphere on which the brane was wrapped. Similarly, the electric flux led to an asymptotically AdS_2 manifold, whose curvature was larger than that of the background AdS_3 space. The sources of the magnetic/electric fluxes were (p, q) strings connecting a $D3$ and a background system, along direction 9 in (2.1). Since a (p, q) string pulls a $D3$ branes, its deforms its surface, and taking the near horizon limit we focus on a "deformed" part of a $D3$ brane, which leads to these kinds of geometries.

The $D7$ brane orthogonally intersecting the $D3$ brane leads, in the near horizon limit, to an $\mathrm{AdS}_4 \times S^4$ brane. This $D7$ brane-embedding solves the DBI equations of motion, though it is obviously not a supersymmetric configuration. This was explicitly checked in [20] by performing a kappa symmetry analysis.[10] We will now show that the *flat* space flux (3.4) solves the equations of motion in an $\mathrm{AdS}_5 \times S^5$ background. Moreover, due to the supersymmetric flat space origin of this flux, it is clear that this configuration is supersymmetric. Note that the second component of the worldvolume flux (3.4) couples the AdS_5 and S^5 components of the metric. To see this explicitly we rewrite the flux using Poincaré coordinates in the AdS_5 part and using the following parametrisation for the five sphere:

$$x_7 = u\cos\theta\cos\varphi\,, \qquad x_8 = u\cos\theta\sin\varphi\,, \qquad x_i = u\sin\theta\,\Omega_i\,, \tag{3.5}$$

with $i = 4,5,6,9$ and $\sum_i \Omega_i^2 = 1$. In these coordinates the metric on the five-sphere becomes

$$ds^2_{S^5} = \cos^2\theta d\varphi^2 + d\theta^2 + \sin^2\theta\Big(d\psi^2 + \sin^2\psi\big(d\xi^2 + \sin^2\xi d\eta^2\big)\Big)\,, \tag{3.6}$$

and the flux (3.4) is given by

$$F = -\cot\alpha\, dx_1\wedge dx_2 + \tan\alpha\, R^4 u\cos^2\theta\, du\wedge d\varphi - \tan\alpha\, R^4u^2\sin\theta\cos\theta d\theta\wedge d\varphi\,. \tag{3.7}$$

Let us now show that the configuration given above solves the DBI action for the $D7$ probe in $\mathrm{AdS}_5 \times S^5$. The calculation simplifies greatly in cartesian coordinates, in which the metric is given by

$$ds^2 = R^2u^2\Big(-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2\Big) + \frac{1}{R^2u^2}\Big(dx_4^2 + \cdots + dx_9^2\Big)\,, \tag{3.8}$$

with $u^2 = (x_4^2 + \cdots + x_9^2)/R^4$. Further simplifications are obtained by going to the static gauge,

$$\sigma_i = x_i\,, \qquad \text{where } i = 0,1,2,4,5,6,7,8\,. \tag{3.9}$$

By examining the Wess-Zumino term it is clear that the $D7$ brane with the above embedding does not couple to the background RR-flux. In this case the general DBI equations of motion given in (A.21) and (A.22) reduce to

$$\mathcal{K}_\mu = 0\,, \qquad \partial_i(\sqrt{-\det M}\theta^{ij}) = 0\,, \tag{3.10}$$

where M_{ij}, G^{ij} and θ^{ij} are defined in (A.25). In the absence of (covariant) flux $\mathcal{F}$ the quantity $\mathcal{K}_\mu$ reduces to the trace of the second fundamental form. In our case of a maximal AdS_4 and a maximal S^4 (given by $x_3 = x_9 = 0$) the second fundamental form vanishes identically. This simplifies the calculation of $\mathcal{K}_\mu$ in the presence of flux. It is easy to see that the non-geometric terms (i.e. flux-dependent terms) in $\mathcal{K}$ cancel separately. One uses

$$\det M = -\frac{1}{u^8R^8\tan^2\alpha}\Big(1 + u^4R^4\tan^2\alpha\Big)^2\,,$$

[10] One might think that taking the near horizon limit of a non-supersymmetric configuration might lead to a supersymmetric configuration. However, in the Penrose limit the $\mathrm{AdS}_4 \times S^4$ brane reduces to a non-supersymmetric $(+,-,3,3)$ brane, and taking a Penrose limit cannot decrease the amount of supersymmetry.

$$G^{00} = -\frac{1}{u^2R^2}\,, \qquad G^{44} = G^{55} = G^{66} = u^2R^2\,,$$
$$G^{11} = G^{22} = \frac{u^2R^2\tan^2\alpha}{1+u^4R^4\tan^2\alpha}\,, \qquad G^{77} = G^{88} = \frac{u^2R^2}{1+u^4R^4\tan^2\alpha}\,, \tag{3.11}$$

with all other components vanishing. The second equation in (3.10) vanishes since θ^{ij} reduces to

$$\theta^{12} = \frac{\tan\alpha}{1+u^4R^4\tan^2\alpha}\,, \qquad \theta^{78} = \frac{-u^4R^4\tan\alpha}{1+u^4R^4\tan^2\alpha}\,, \tag{3.12}$$

with vanishing other components for the flux (3.4).

Finally, we would like to point out that this kind of D-brane, with non-diagonal flux, is not covered by the general analysis of (generalised) calibrations with worldvolume fluxes as studied in [21]. The cases covered in [21] are supersymmetric in the absence of worldvolume flux, while our D-brane is non-supersymmetric if the flux is zero. The non-diagonal flux will also lead to an interesting Penrose limit which we will discuss in section 4.3.

4. Penrose limits

4.1 Penrose limit of the diagonal $D3$ brane: a brane with a pulse

It is usually said that in order to have a nontrivial Penrose limit of a brane in some background, one needs to take the limit along a geodesic which belongs to the brane. This statement is intuitively understandable: in the Penrose limit an infinitesimal region around the geodesics gets zoomed out. Hence, those parts of the brane which are placed at some nonzero distance from the geodesic get pushed off to infinity. However, this reasoning can be circumvented if the distance between the geodesic and the brane is determined by free parameters of the solution. In that case one can take the Penrose limit along a geodesic that *does not* belong to the brane, as long as the parameter labeling the brane in a family of solutions is appropriately scaled.

For example, let us consider the family of solutions corresponding to two intersecting D-branes and let us take the Penrose geodesic to lie on one of the two branes. Then the Penrose limit of the other brane can be nontrivial if, while taking the Penrose limit of the target space metric, we simultaneously scale the angle between the two branes to zero. It should be emphasised that the final configuration obtained in this way is different from the one which is obtained by first sending the angle to zero and then taking the Penrose limit of the metric.

To see how these ideas work in practice, we consider in the next two sections the Penrose limit of the family of diagonal $D3$ branes given in (2.8) with the Penrose geodesic taken to be a generic null geodesic that mixes the AdS, sphere and torus parts and does not belong to the brane worldvolume. In the first section, 4.1.1, we work out the Penrose limit of the background, obtained by using a generic geodesic. In the second section, 4.1.2, we restrict the analysis to a subclass of geodesics that do not wind on the torus (and do not belong to the brane worldvolume) and apply a Penrose limit on the brane while simultaneous taking the angle to zero. We end up with a brane which is the same as the one which we would get if we would first have sent the angle to zero, except that there is now a null pulse propagating on the brane worldvolume.

4.1.1 Penrose limit of the $\mathrm{AdS}_3 \times S^3 \times T^4$ background

Let us consider a null geodesic which mixes the AdS, sphere, and torus parts, extends in the subspace (t, u, ψ, x_2, x_3) in Poincare coordinates (2.6) and is placed at a constant value of transverse coordinates[11]

$$x_1 = x_4 = x_5 = 0\,, \qquad \xi = \frac{\pi}{2}\,, \qquad \eta = 0\,. \tag{4.1}$$

As far as the Penrose limit of the metric goes, these values of the transverse coordinates are irrelevant. The choice which we have made here is such that the analysis of the Penrose limit of the brane in the next section is simplified. To find the geodesic we make use of the following conserved quantities (dots denote derivatives with respect to the affine parameter τ):

$$R^2 u^2 \dot{t} = E\,, \qquad R^2 \dot{\psi} = l\,, \qquad R_t^2 \dot{x}_2 = p_2\,, \qquad R_t^2 \dot{x}_3 = p_3\,. \tag{4.2}$$

We also use the null condition $ds^2 = 0$. The geodesic is given by

$$\begin{aligned} u &= \frac{E}{L}\sin\left(\frac{L}{R^2}\tau\right), \qquad & t &= -\frac{L}{E}\cot\left(\frac{L}{R^2}\tau\right), \qquad & \psi &= \frac{l}{R^2}\tau\,, \\ x_2 &= \frac{p_2}{R_t^2}\tau\,, & x_3 &= \frac{p_3}{R_t^2}\tau\,, & L^2 &\equiv l^2 + \frac{R^2}{R_t^2}(p_2^2 + p_3^2)\,. \end{aligned} \tag{4.3}$$

Next we go to an adapted coordinate system in which one of the new coordinates (namely the coordinate $\tilde{u}$) is identified with the affine parameter along the null geodesic (i.e. $\tilde{u} = \tau$). All other coordinates are chosen in such a way that the geodesic is located at the origin in the remaining directions. More precisely, we introduce new coordinates

$$(\tilde{u}, \tilde{v}, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{x}_i)\,, \qquad \text{with } i = 1, 2, 3, 4, 5\,. \tag{4.4}$$

We require that in the new coordinates the metric components $g_{\tilde{u}\tilde{u}} = g_{\tilde{u}\tilde{x}_2} = g_{\tilde{u}\tilde{x}_3} = 0$ and $g_{\tilde{u}\tilde{v}} = R^2$. To find a set of coordinates which fulfills these requirements we use the ansatz

$$u = \frac{E}{L}\sin\left(\frac{L}{R^2}\tilde{u}\right) + f_1(\tilde{v}, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{x}_i)\,, \qquad \psi = \frac{l}{R^2}\,\tilde{u} + f_2(\tilde{v}, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{x}_i)\,, \qquad \text{etc.} \tag{4.5}$$

One possible choice for an adapted coordinate system with these properties is given in the appendix, see equation (A.1). By writing the metric in this adapted coordinate system (see (A.2)) one can see that in order to obtain a finite result for the metric in the $\lambda \to 0$ limit, we have to scale the tilded coordinates and the parameters R and R_t according to

$$\begin{aligned} &R \to \lambda^{-1} R\,, \qquad && R_t \to \lambda^{-1} R_t\,, \qquad && \tilde{u} \to \lambda^{-2}\tilde{u}\,, \qquad && \tilde{v} \to \lambda^4 \tilde{v}\,, \\ &\tilde{\phi} \to \lambda\,\tilde{\phi}\,, && \tilde{\xi} \to \lambda\tilde{\xi}\,, && \tilde{\eta} \to \lambda\tilde{\eta}\,, && \tilde{x}_i \to \lambda\tilde{x}_i\,, \qquad \text{with } i = 1, 2, 3, 4, 5\,. \end{aligned} \tag{4.6}$$

[11] The analysis of a geodesic which also involves the direction ξ is more involved but leads to a similar conclusion, namely that for the generic brane (2.8) there is no null geodesics which belongs to its worldvolume. The geodesics we consider here are slight generalisations of the geodesics analysed in [22].

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Recall that we have chosen to work with dimensionless coordinates; hence we take a Penrose limit by scaling only the dimensionless quantities and keeping α' fixed. In the limit (4.6) the metric (A.2) in adapted coordinates reduces to metric of the following form (expressed in *Rosen coordinates*):

$$ds^2 = 2R^2 d\tilde{u} d\tilde{v} + \sum_{ij} C_{ij}(\tilde{u})\, dy^i dy^j \tag{4.7}$$

where y^i denotes all coordinates apart from $\tilde{u}$ and $\tilde{v}$, and C_{ij} is a symmetric *non-diagonal* matrix. To put this metric into more familiar Brinkman coordinates one needs to follow a procedure outlined in [22]. For us however, the explicit form of the general metric (A.2) in Brinkman coordinates will not be relevant, so we do not perform these coordinate transformations here.[12]

On the other hand, one can check that in the $p_2 = p_3 = 0$ case, which will be used in the next section, equation (A.2) reduces to a standard Hpp-wave in Brinkman coordinates, after the change of coordinates given by (A.3). The wave metric is given by

$$\begin{aligned} ds^2 &= 2dx^+dx^- - \frac{4L^2}{R^4}\Big(dy_1^2 + dy_2^2 + dz_1^2 + dz_2^2\Big)\big(dx^+\big)^2 + dy_1^2 + dy_2^2 + \\ &\quad + dz_1^2 + dz_2^2 + R_t^2\Big(d\tilde{x}_2^2 + \cdots + d\tilde{x}_5^2\Big)\,, \end{aligned} \tag{4.9}$$

and as usual, due to the NS-NS flux, the manifest $\mathrm{SO}(6) \times R^4$ symmetry is broken to $\mathrm{SO}(3)_y \times \mathrm{SO}(3)_z \times R^4_{\tilde{x}}$.

4.1.2 Penrose limit of the $D3$ brane

We are now ready to discuss the Penrose limit of the $D3$ brane (2.8). Here we will use the following parametrisation of the coordinates (x_6, x_7, x_8, x_9) in (2.5):

$$x_7 = u\cos\psi\,, \quad x_8 = u\sin\psi\sin\xi\,, \quad x_6 = u\sin\psi\sin\xi\cos\eta\,, \quad x_9 = u\sin\psi\sin\xi\sin\eta\,. \tag{4.10}$$

Note that the parametrisation of the Poincaré coordinates which we use here is different from the one used in equation (2.4). Other parameterisations correspond to different relative orientations of the Penrose geodesic with respect to the brane worldvolume. In principle these other choices could lead to different Penrose limits, but it turns out that all these limits share the qualitative characteristics of the one which we discuss. Notice also that if $\alpha \neq 0$ the geodesic described by (4.1) and (4.3) cannot belong to the $D3$ brane for any value of the parameters E, l, p_2 and p_3, while if $\alpha = 0$ then for $p_2 = p_3 = 0$ and arbitrary E, l the geodesic belongs to the brane worldvolume.

[12] For generic values of the momenta p_2 and p_3 one is not automatically guaranteed that the metric (4.7) describes a Lorenzian symmetric (Cahen-Wallach) space, i.e. that it is one of the Hpp-waves. For that one needs to check that the matrix A_{ij} in the metric written in Brinkman coordinates,

$$ds^2 = 2dx^+dx^- + \sum_{ij} A_{ij}(x^-)x^i x^j (dx^-)^2 + dx_i^2 \tag{4.8}$$

is a constant matrix. In our case a change of coordinates from Rosen to Brinkman is more involved due to the fact that C_{ij} is non-diagonal.

To obtain the Penrose limit of the $D3$ brane we rewrite equations (2.8), describing the embedding of $D3$ brane, in the adapted coordinate system (A.1),

$$\begin{aligned}
\tilde{x}_1 &= \tilde{x}_4 = \tilde{x}_5 = \tilde{\eta} = 0\,, \\
\frac{p_2}{R_t^2}\,\tilde{u} + \tilde{x}_2 &= \frac{E}{L}\,\tan\alpha\,\sin\left(\frac{L}{R^2}\tilde{u}\right)\cos\left(\frac{l}{R^2}\tilde{u} + \tilde{\phi} - \frac{p_2}{l}\tilde{x}_2 - \frac{p_3}{l}\tilde{x}_3\right), \\
\frac{p_3}{R_t^2}\,\tilde{u} + \tilde{x}_3 &= \frac{E}{L}\,\tan\alpha\,\sin\left(\frac{L}{R^2}\tilde{u}\right)\sin\left(\frac{l}{R^2}\tilde{u} + \tilde{\phi} - \frac{p_2}{l}\tilde{x}_2 - \frac{p_3}{R^2}\tilde{x}_3\right)\sin\left(\frac{\pi}{2} + \tilde{\xi}\right). \qquad (4.11)
\end{aligned}$$

We see that when $p_2 \neq 0$ and $p_3 \neq 0$ an application of the scaling (4.6) to the previous equations leads to equations which can be satisfied only by $\tilde{u} = 0$. Since $\tilde{u}$ is a time-like coordinate, this means that in the Penrose limit the $D3$ brane becomes instantonic.

To obtain a more physical result, consider the $p_2 = p_3 = 0$ geodesic and scale the angle α such that

$$\tan\alpha \to \lambda\,\tan\alpha\,. \qquad (4.12)$$

In other words: as we start zooming out the region near the geodesic (which does not belong to the brane) we simultaneously bring the brane closer and closer to the geodesic by sending the angle to zero. In this case equations (4.11) reduce in the limit to

$$\begin{aligned}
y_2 &= z_2 = \tilde{x}_4 = \tilde{x}_5 = 0\,, \\
\tilde{x}_2 &= \tan\,\alpha\,\frac{E}{l}\sin\left(\frac{2l}{R^2}x^+\right)\cos\left(\frac{2l}{R^2}x^+\right), \\
\tilde{x}_3 &= \tan\,\alpha\,\frac{E}{l}\sin\left(\frac{2l}{R^2}x^+\right)\sin\left(\frac{2l}{R^2}x^+\right). \qquad (4.13)
\end{aligned}$$

We see that the Penrose limit of the diagonal D3-brane produces a brane with orientation $(+,-,1,1,0)$, i.e. a $D3$ brane that extends in the directions x^+, x^-, y_1, z_1. In addition, the brane carries a null wave-like excitation in the directions $\tilde{x}_2(x^+)$ and $\tilde{x}_3(x^+)$.[13] These kind of wave excitations have been studied recently in [8]. Note that the DBI action allows for an arbitrary profile of transverse null-like excitations.[14] In our case, however, the precise form of these worldvolume waves carries information about the position of the brane with respect to the geodesic before the limit has been taken. Note also that if we *first* take the $\alpha \to 0$ limit (and adjust the geodesic in such a way that it lies on the worldvolume of the brane) and then take the Penrose limit, we would end up with the $(+,-,1,1,0)$ brane with no wave on its worldvolume.

4.2 Penrose limit of the diagonal $D4$ brane: an oblique brane

In this section we will study the Penrose limit of the diagonal $D4$ brane (2.40). We restrict to this simpler case, rather than discussing the generic $D4$ brane (2.36), since we are interested in taking a Penrose limit along a geodesic that winds between the two

[13] We should emphasise that the wave which is obtain here is a genuine wave, i.e. it is not a mere coordinate artifact, like the one which appeared in [23].

[14] It was shown in [8] that the DBI waves are supersymmetric, *exact* solutions of the DBI action to all orders in α'.

submanifolds and *unlike* in the previous section, *belongs* to the brane. The analysis of the geodesics (which do or do not wind between submanifolds) but do not belong to the brane worldvolume leads to Penrose limits similar to the one which was already discussed.

So let us consider a geodesic which winds diagonally between the two big circles of two three-spheres in the target space (2.34) and sits at the origin of the AdS_3 space. The product of two big circles is just an (orthogonal) torus, and we focus on a geodesic which diagonally winds in this torus. If we use the coordinates (A.20) for the three spheres, the geodesic under the angle ω (with respect to one of the big circles) can be written as

$$\eta \equiv a\cos\omega\eta_1 + b\sin\omega\eta_2 = \tau\,, \quad \tilde{\eta} \equiv -a\sin\omega\,\eta_1 + b\cos\omega\,\eta_2 = c = \text{const.}\,, \tag{4.14}$$
$$\rho = 0\,, \quad \lambda = g = \text{const.} \quad \psi_i = \xi_i = \frac{\pi}{2} \quad (i=1,2) \quad \left(a = \frac{\sqrt{N_F^{(2)}}}{R}\,, \quad b = \frac{\sqrt{N_F^{(1)}}}{R}\right).$$

Going through the standard procedure [24] of introducing the light cone coordinates $x^\pm$ and a set of new coordinates $(x_1, x_2, y_1, y_2, z_1, z_2, \phi, \tilde{\lambda})$ as

$$\begin{aligned} \tau &= x^+ + \frac{1}{R^2}x^-\,, \quad \eta = x^+ - \frac{1}{R^2}x^-\,, \quad \tilde{\eta} = c + \frac{1}{R}\phi\,, \quad \lambda = g + \tilde{\lambda}\,, \quad \rho = \frac{\tilde{r}}{R}\,, \\ \psi_1 &= \frac{\pi}{2} - \frac{y_1}{\sqrt{N_F^{(1)}}}\,, \quad \xi_1 = \frac{\pi}{2} - \frac{y_2}{\sqrt{N_F^{(1)}}}\,, \quad \psi_2 = \frac{\pi}{2} - \frac{z_1}{\sqrt{N_F^{(2)}}}\,, \quad \xi_2 = \frac{\pi}{2} - \frac{z_2}{\sqrt{N_F^{(2)}}}\,, \end{aligned} \tag{4.15}$$

with $\tilde{r}^2 = x_1^2 + x_2^2$, and scaling all parameters to infinity in such a way that the parameters a and b are kept fixed, we obtain the following pp-wave metric and three-form flux

$$\begin{aligned} ds^2 &= -4dx^+dx^- - \left(x_1^2 + x_2^2 + \frac{\cos^2\omega}{a^2}(y_1^2+y_2^2) + \frac{\sin^2\omega}{b^2}(z_1^2+z_2^2)\right)(dx^+)^2 + \\ &\quad + d\vec{x}^2 + d\vec{y}^2 + d\vec{z}^2 + d\phi^2 + d\lambda^2\,, \\ H &= 2\,\tilde{r}dx^+ \wedge d\tilde{r} \wedge d\sigma + 2\frac{\sin\omega}{b}\,dx^+ \wedge dy_1 \wedge dy_2 + 2\frac{\cos\omega}{a}dx^+ \wedge dz_1 \wedge dz_2\,. \end{aligned} \tag{4.16}$$

This is just the standard Nappi-Witten background with different values of NS flux in different two planes, and an isometry group (in the space-like directions) given by the product $G = \mathrm{SO}(2)_x \times \mathrm{SO}(2)_y \times \mathrm{SO}(2)_z \times R^2_{\lambda,\phi}$. The previous discussion holds for arbitrary $N_F^{(1)}$ and $N_F^{(2)}$ but in order to study (2.40) we furthermore restrict to the case $N_F^{(1)} = N_F^{(2)} \equiv N_F$.

Before we apply the Penrose limit to (2.40), note that when the parameters g, c and ω take the values

$$g = R\ln(\tan\alpha)\,, \qquad c = 0\,, \qquad \omega = \frac{\pi}{4}\,, \tag{4.17}$$

the geodesic (4.14) satisfies the embedding equations of the brane and hence belongs to its worldvolume. Applying the scaling (4.15) to the equation (2.40), with the values of the parameters given above, one obtains the embedding equations for the $D4$ brane in the pp-wave (4.16),

$$x_2 = y_1 = z_1 = 0\,, \qquad \phi = 0\,, \qquad \frac{1}{N_F}(z_2^2 - y_2^2) = \frac{\lambda}{R}\,. \tag{4.18}$$

Since λ does not scale, while N_F and R scale to infinity in such a way that $a = \text{const.}$, the third equation reduces in the limit to $\lambda = 0$. In order to obtain a more interesting Penrose limit we will scale λ to zero faster than $1/R$. In this case the third equation reduces to

$$z_2 = \pm y_2 \,. \tag{4.19}$$

We see that the resulting brane (4.18), (4.19) has orientation $(+,-,1,1/2,1/2,1)$. Here we have used the isometry group G to group the brane worldvolume directions; the notation $(1/2,1/2)$ means that the brane is diagonally embedded between y_2 and z_2. Hence we have obtained a brane that is diagonally (i.e. in an oblique way) embedded between the $\mathrm{SO}(2)_y$ and $\mathrm{SO}(2)_z$ subspaces. This type of brane was first observed in [9], but their AdS origin was not known so far.

Note however, that the scaling of λ which we have used leads to an effective compactification in the limit (from ten to nine dimensions). Although there are no conceptual reasons why one could not take this kind of Penrose limit, this is not how the Penrose limit is usually taken. By considering a more complicated geodesic, one can produce oblique branes from (2.40) without having to compactify space in the limit. However, this computation is more tedious and without an additional physical outcome, so we present it in the appendix.

4.3 Penrose limit of the brane with flux: a brane with null flux

Let us now consider the Penrose limit of the $\mathrm{AdS}_4\times S^4$, $D7$ brane with diagonal worldvolume flux. We have seen in the previous section that in order to obtain a nontrivial Penrose limit of the brane under an angle, we were forced to bring the brane closer and closer to the geodesic while taking the limit. Inspired by the "equivalence" of geometry and worldvolume fluxes, as in section 3, we expect that a similar story will repeat itself in the case of the brane with flux (3.4). There is however a slight difference. In the purely geometrical setup, it is clear when the geodesic lies along the brane, and hence when the "standard" Penrose limit (without any additional scaling of a parameter of the solution) can be taken. In the case of a non-vanishing flux the "standard" geodesics should, as we will see, be taken in a direction *orthogonal* to the flux.

This feature can roughly be understood already in the context of $3+1$ dimensional electromagnetism in flat space. Imagine that a magnetic flux $F_{12}\, dx^1 \wedge dx^2$ is turned on. Boosting with the velocity of light in the direction x^3 results in zero electric and magnetic fields (since we "contract" the length of the magnetic field pointing in the third direction). Boosting in the directions one or two, on the other hand, leads to infinite electric and magnetic fields.[15] Taking the Penrose limit in curved space is a more complicated procedure since it involves both boosting and additional rescaling, so let us now check how this simple flat-space analysis generalises to the full curved space.

[15]Recall that for a general Lorentz transformation from the system K to the system K' moving with velocity $\vec{v}$ relative to K, the transformation of the electric and magnetic fields is given by

$$\begin{aligned}
\vec{E}' &= \gamma(\vec{E} + \vec{\beta}\times\vec{B}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}\cdot(\vec{\beta}\cdot\vec{E})\,,\\
\vec{B}' &= \gamma(\vec{B} - \vec{\beta}\times\vec{E}) - \frac{\gamma^2}{\gamma+1}\vec{\beta}\cdot(\vec{\beta}\cdot\vec{B})\,.
\end{aligned} \tag{4.20}$$

Let us consider a geodesic that winds along the big circle parametrised by η and located at $\theta = \psi = \xi = \pi/2$ in S^5 in (3.6), and is located at the origin $\rho = 0$ of AdS_5 in global coordinates. In Poincaré coordinates this geodesic is lying in the $4-5$ plane. As usual, we now introduce new coordinates and perform a rescaling as in [24],

$$\begin{aligned} &\tau = x^+ + \frac{x^-}{R^2}\,, \quad \eta = x^+ - \frac{x^-}{R^2}\,, \quad \rho = \frac{z}{R}\,, \quad (z^2 = z_1^2 + \cdots + z_4^2)\,, \\ &\theta = \frac{\pi}{2} + \frac{\tilde{y}}{R}\,, \quad \psi = \frac{\pi}{2} + \frac{y_3}{R}\,, \quad \xi = \frac{\pi}{2} + \frac{y_4}{R}\,, \quad \varphi \to \varphi \quad (\tilde{y}^2 = y_1^2 + y_2^2)\,. \end{aligned} \tag{4.21}$$

In this limit the geometrical part of the $D7$ brane reduces to a brane with the embedding $(+,-,3,3)$.

To see the effect of the Penrose limit on the flux, let us first rotate the flux (3.4) in the $4-7$ and $5-8$ planes, so that it is placed under arbitrary angles ν_1 and ν_2 with respect to the above geodesic. For $\nu_1 = \nu_2 = 0$ the flux is in the plane $7-8$, i.e. it is orthogonal to the geodesic. We will denote the coordinates in section 3 with primes and the coordinates used in this section without primes. We have

$$\begin{aligned} x'_7 &= \cos\nu_1\, x_7 + \sin\nu_1\, x_4\,, \quad x'_4 = -\sin\nu_1\, x_7 + \cos\nu_1\, x_4\,, \\ x'_8 &= \cos\nu_2\, x_8 + \sin\nu_2\, x_5\,, \quad x'_5 = -\sin\nu_2\, x_8 + \cos\nu_2\, x_5\,, \quad x'_i = x_i \text{ (all other)}\,, \end{aligned} \tag{4.22}$$

and the flux (3.4) becomes

$$\begin{aligned} F &= -\cot\alpha\, dx'_1 \wedge dx'_2 + \tan\alpha\, dx'_7 \wedge dx'_8 \\ &= -\cot\alpha\, dx_1 \wedge dx_2 + \tan\alpha \cos\nu_1 \cos\nu_2\, dx_7 \wedge dx_8 + \\ &\quad + \tan\alpha\Big(\sin\nu_1 \cos\nu_2\, dx_4 \wedge dx_8 + \cos\nu_1 \sin\nu_2\, dx_7 \wedge dx_5 + \sin\nu_1 \sin\nu_2\, dx_4 \wedge dx_5\Big)\,. \end{aligned} \tag{4.23}$$

It is easy to see that the leading terms in the $1/R$ expansion of the flux are given by the following expression

$$\begin{aligned} F &= -\cot\alpha\, d\left(\frac{z_1}{\cos x^+}\right) \wedge d\left(\frac{z_2}{\cos x^+}\right) + \\ &\quad + \tan\alpha \cos\nu_1 \cos\nu_2 \Big(\tilde{y} \sin x^+ \cos x^+\, dx^+ \wedge d\varphi - \cos^2 x^+\, d\tilde{y} \wedge d\varphi\Big) + \\ &\quad + R\, \tan\alpha\tilde{y}\Big(\sin\nu_1 \cos\nu_2 \cos 2x^+ + \cos\nu_1 \sin\nu_2 \sin 2x^+\Big) \cos x^+\, \cos\varphi\, d\varphi \wedge dx^+ - \\ &\quad - R\, \tan\alpha \sin\nu_1 \sin\nu_2 \cos x^+\, dx^+ \wedge dz_4 + \mathcal{O}\left(\frac{1}{R}\right)\,, \end{aligned} \tag{4.24}$$

where $z_4 \equiv z\Omega_4$ and the angular parameter Ω_4 is defined in (3.5). We see that if the Penrose geodesic is taken to be orthogonal to the flux (i.e. $\nu_1 = \nu_2 = 0$) then there is no need to perform any additional scaling of the above flux. If however $\nu_1 \neq 0$ and $\nu_2 \neq 0$, then for the Penrose limit to be well-defined we have to simultaneously send $\nu_1 \to 0$ and $\nu_2 \to 0$ while taking the limit. As in the purely geometrical case, taking the Penrose limit and rescaling the parameters are operations that do not commute. The flux obtained by taking the Penrose limit along the orthogonal geodesic is *different* from the one obtained by looking at a geodesic under an angle with the angle rescaled to zero while taking the limit.

The case of an orthogonal flux has already appeared in the literature [20] in the case of the Penrose limit of the $\mathrm{AdS}_4 \times S^2$ brane which wraps a maximal S^2 and carries magnetic flux on the S^2. In this case the Penrose limit was taken along the S^1 in S^2 i.e. parallel to the flux, with the expected consequence that the flux had to be rescaled in order to obtain a finite Penrose limit.

Finally let us conclude this section with the observation that the brane with flux which was obtained by considering the previous Penrose limit is of the form discussed recently in [25]. In flat space a D-brane with a null flux is T-dual to a D-brane with a null pulse, like the one which was obtained in the previous section. In the pp-wave this duality is less manifest, since T-duality changes the background as well. Throughout this paper we have tried to trace how these dual configurations are realised in flat space, AdS and finally pp-wave spacetimes.

5. Discussion

In this paper we have presented various types of D-branes which mix target space manifolds in a geometrical way or through diagonal worldvolume fluxes. Our consideration was classical and based on a probe brane analysis. It would be interesting to extend our analysis to the supergravity regime, taking into account the back-reaction of the D-branes as in [23]. It would also be desirable to obtain a description of these branes (at least in the Penrose limit) using the covariant approach of [26].

Our initial motivation for studying diagonal D-branes in product spaces was to understand the construction of boundary states for these branes. This problem, however, turned out to be hard, essentially due to the fact that some of the target space symmetries are completely broken by the brane; there is no diagonal current preserved by the boundary conditions. This of course does not mean that the brane is inconsistent. The consistency of the theory only requires the diagonal part of the Virasoro algebra to be preserved. Moreover, due to the way these branes were constructed, we are guaranteed that they are consistent. However, it is not clear to us at the moment how one should implement the properties of these branes at the level of boundary states. One possible way in which one might be able to improve the understanding of some of the features of diagonal branes is by a classical analysis of their spectrum (as in [27]). An alternative way, mentioned in the introduction, is to study these branes using the dCFT/AdS correspondence [4]. Finally, a construction of the boundary states in the Penrose limit might also help to understand these branes in AdS geometries.

Acknowledgments

We would like to thank Matthias Blau, Jerome Gauntlett, George Papadopoulos, Kasper Peeters and Volker Schomerus for useful discussions.

A. Technical details

A.1 Technical details about the Penrose limit of $\mathbf{AdS}_3 \times S^3 \times T^4$

In this section we give some technical details about the calculation in section 4.1.1. One possible choice for the adapted coordinate system which fulfills the requirements listed around (4.4) is given by

$$\begin{aligned}
u &= \frac{E}{L}\sin\left(\frac{L}{R^2}\tilde{u}\right), \qquad \psi = \frac{l}{R^2}\,\tilde{u} + \tilde{\phi} - \frac{p_2}{l}\,\tilde{x}_2 - \frac{p_3}{l}\,\tilde{x}_3\,, \\
t &= -\frac{L}{E}\cot\left(\frac{L}{R^2}\tilde{u}\right) - \frac{R^2}{E}\,\tilde{v} + \frac{l}{E}\,\tilde{\phi}\,, \qquad x_2 = \frac{p_2}{R_t^2}\,\tilde{u} + \tilde{x}_2\,, \qquad x_3 = \frac{p_3}{R_t^2}\,\tilde{u} + \tilde{x}_3\,, \\
\xi &= \frac{\pi}{2} + \tilde{\xi}\,, \qquad \eta = \tilde{\eta}\,, \qquad x_i = \tilde{x}_i\,, \qquad \text{with } i = 1,4,5\,.
\end{aligned} \tag{A.1}$$

The metric (2.6) in adapted coordinates becomes

$$\begin{aligned}
ds^2 =&\ 2R^2\, d\tilde{u}d\tilde{v} - \frac{R^6}{L^2}\sin^2\left(\frac{L}{R^2}\tilde{u}\right) d\tilde{v}^2 + \left(-\frac{l^2}{L^2}R^2\sin^2\left(\frac{L}{R^2}\tilde{u}\right) + R^2\right) d\tilde{\phi}^2 + \\
&+ 2R^4\frac{l}{L^2}\sin^2\left(\frac{L}{R^2}\tilde{u}\right) d\tilde{v}d\tilde{\phi} - 2\frac{p_2}{l}R^2 d\tilde{x}_2 d\tilde{\phi} - 2\frac{p_3}{l}R^2 d\tilde{x}_3 d\tilde{\phi} + \\
&+ \left(R_t^2 + \frac{p_2^2}{l^2}R^2\right) d\tilde{x}_2^2 + \left(R_t^2 + \frac{p_3^2}{l^2}R^2\right) d\tilde{x}_3^2 + \frac{2p_2p_3R^2}{l^2} d\tilde{x}_2 d\tilde{x}_3 + \\
&+ R_t^2\left(d\tilde{x}_4^2 + d\tilde{x}_5^2\right) + R^2\sin^2(\psi)\, ds^2(S^2) + \frac{R^2E^2}{L^2}\sin^2\left(\frac{L}{R^2}\tilde{u}\right) d\tilde{x}_1^2\,.
\end{aligned} \tag{A.2}$$

Next we apply the scaling (4.6) to the metric (A.2) and set $p_2 = p_3 = 0$. In order to rewrite the metric in Brinkman coordinates, one then uses the following coordinate transformations [22]:

$$\begin{aligned}
\tilde{u} &= 2x^+\,, \qquad \tilde{v} = \frac{1}{2R^2}\left(x^- + \frac{1}{2}\sum_{i=1}^{4} a_i^{-1}\partial_{x^+}(a_i)(x_i)^2\right), \qquad (x_i \equiv y_1, y_2, z_1, z_2)\,, \\
y_1 &= R\cos\left(\frac{2L}{R^2}x^+\right)\tilde{\varphi} \equiv a_1(x^+)\tilde{\varphi}\,, \qquad y_2 = \frac{RE}{L}\sin\left(\frac{2L}{R^2}x^+\right)\tilde{x}_1 \equiv a_2(x^+)\tilde{x}_1\,, \\
z_1 &= R\sin\left(\frac{2L}{R^2}x^+\right)\tilde{\xi} \equiv a_3(x^+)\tilde{\xi}\,, \qquad z_2 = R\sin\left(\frac{2L}{R^2}x^+\right)\tilde{\eta} \equiv a_4(x^+)\tilde{\eta}\,,
\end{aligned} \tag{A.3}$$

while the remainder of the coordinates remain unchanged. The resulting metric is given in equation (4.9).

A.2 Technical details about the Penrose limit of the diagonal D4-brane

Here we present a Penrose limit of the diagonal $D4$ brane (2.40) which is different from the one given in section 4.2. The resulting brane is again an oblique brane, but the target space is ten-dimensional, unlike the one in (4.18) and (4.19). Let us consider the following geodesic in Poincaré coordinates,

$$u = \frac{E}{L}\sin\left(\frac{L}{R^2}\tau\right), \qquad t = -\frac{L}{E}\cot\left(\frac{L}{R^2}\tau\right),$$

$$\xi_a \equiv \xi_1 + \xi_2 = \tau\,, \qquad \xi_b \equiv -\xi_1 + \xi_2 = c_\xi\,. \tag{A.4}$$
$$\psi_1 = \psi_2 = \frac{\pi}{2}\,, \qquad \eta_1 = \eta_2 = c_\eta\,, \qquad \lambda = g\,, \qquad x_1 = c_x\,, \qquad a = b = \frac{\sqrt{N_F}}{R} = \sqrt{2}\,.$$

It is easy to see that for $c_x = c_\eta = c_\xi = g = 0$, this geodesic belongs to the worldvolume of the brane. It winds along a diagonal curve ($\xi_1 = \xi_2$) which is "orthogonal" to the other diagonal direction ($\eta_1 = \eta_2$) that the brane wraps as well. In what follows we will make a restriction to this kind of geodesic.

Next we go to coordinate system adapted to this geodesic, following a procedure similar to the one in section 4.1.

$$(u, t, x_1, \psi_1, \psi_2, \xi_a, \xi_b, \eta_1, \eta_2) \to (\tilde{u}, \tilde{v}, \tilde{x}_1, \tilde{\psi}_1, \tilde{\psi}_2, \xi_1, \tilde{\xi}_a, \tilde{\eta}_a, \tilde{\eta}_b)\,, \tag{A.5}$$

where the new coordinates are given by

$$\begin{aligned}
u &= \frac{E}{L}\sin\left(\frac{L}{R^2}\tilde{u}\right), & t &= -\frac{L}{E}\cot\left(\frac{L}{R^2}\tilde{u}\right) - \frac{R^2}{E}\tilde{v} + \frac{L}{E}\tilde{\xi}_a\,,\\
\xi_a &= \frac{L}{R^2}\tilde{u} + \tilde{\xi}_a\,, & \xi_b &= \tilde{\xi}_b\\
\psi_1 &= \frac{\pi}{2} + \tilde{\psi}_1\,, & \psi_2 &= \frac{\pi}{2} + \tilde{\psi}_2\,,\\
\tilde{\lambda} &= \lambda\,, & \tilde{x}_1 &= x_1\,.
\end{aligned} \tag{A.6}$$

The $\mathrm{AdS}_3 \times S^3 \times S^3 \times R$ metric written in these coordinates is given by

$$\begin{aligned}
ds^2 = {} & \left(\frac{L}{R}\right)^2 \left(\cos^2\tilde{\psi}_1 + \cos^2\tilde{\psi}_2 - 2\right)\frac{d\tilde{u}^2}{2} + 2R^2 d\tilde{u}d\tilde{v} - \frac{R^6}{L^2}\sin^2\left(\frac{L}{R^2}\tilde{u}\right)d\tilde{v}^2 +\\
& + L(-2 + \cos^2\tilde{\psi}_1 + \cos^2\tilde{\psi}_2)d\tilde{u}d\tilde{\xi}_a + L(\cos^2\tilde{\psi}_2 - \cos^2\tilde{\psi}_1)d\tilde{u}d\tilde{\xi}_b +\\
& + R^2\left(-\sin^2\left(\frac{L}{R^2}\tilde{u}\right) + \frac{\cos^2\tilde{\psi}_1 + \cos^2\tilde{\psi}_2}{2}\right)d\tilde{\xi}_a^2 + \frac{R^2}{2}(\cos^2\tilde{\psi}_1 + \cos^2\tilde{\psi}_2)d\tilde{\xi}_b^2 +\\
& + R^2\left(\cos^2\tilde{\psi}_2 - \cos^2\tilde{\psi}_1\right)d\xi_a d\xi_b + 2R^2\left(d\tilde{\psi}_1^2 + d\tilde{\psi}_2^2\right) +\\
& + 2R^2\cos^2\tilde{\psi}_1\sin^2\frac{1}{2}\left(\frac{L}{R^2}\tilde{u} + \tilde{\xi}_a - \tilde{\xi}_b\right)d\tilde{\eta}_1^2 +\\
& + 2R^2\cos^2\tilde{\psi}_2\sin^2\frac{1}{2}\left(\frac{L}{R^2}\tilde{u} + \tilde{\xi}_a + \tilde{\xi}_b\right)d\tilde{\eta}_2^2 +\\
& + R^2\frac{E^2}{L^2}\sin^2\left(\frac{L}{R^2}\tilde{u}\right)d\tilde{x}_1^2 + \frac{R^4}{L}\sin^2\left(\frac{L\tilde{u}}{R^2}\right)d\tilde{v}d\tilde{\xi}_a + d\tilde{\lambda}^2\,.
\end{aligned} \tag{A.7}$$

We then apply the following rescaling of the metric, using $\Lambda \to 0$,

$$\begin{aligned}
&R \to \Lambda^{-1}R\,, \qquad \tilde{u} \to \Lambda^{-2}\tilde{u}\,, \qquad \tilde{v} \to \Lambda^4\tilde{v}\,, \qquad \tilde{\lambda} \to \tilde{\lambda}\\
&\tilde{x}_i \to \Lambda\,\tilde{x}_i\,, \qquad (\tilde{x}_i = \tilde{\psi}_1, \tilde{\psi}_2, \tilde{\eta}_1, \tilde{\eta}_2, \tilde{\xi}_a, \tilde{\xi}_b, \tilde{x}_1)\,.
\end{aligned} \tag{A.8}$$

Note that the coordinate $\tilde{\lambda}$ does not get scaled in the limit. Under this scaling the metric becomes

$$\begin{aligned} ds^2 =& \frac{L^2}{R^4}(-\tilde{\psi}_1^2 - \tilde{\psi}_2^2)\frac{d\tilde{u}^2}{2} + 2R^2 d\tilde{u}d\tilde{v} + R^2\cos^2\left(\frac{L}{R^2}\tilde{u}\right)d\tilde{\xi}_a^2 + R^2 d\tilde{\xi}_b^2 + \\ &+2R^2(d\tilde{\psi}_1^2 + d\tilde{\psi}_2^2) + 2R^2\sin^2\left(\frac{L}{2R^2}\tilde{u}\right)(d\tilde{\eta}_1^2 + d\tilde{\eta}_2^2) + \\ &+R^2\frac{E^2}{L^2}\sin^2\left(\frac{L}{R^2}\tilde{u}\right)d\tilde{x}_1^2 + d\tilde{\lambda}^2 . \end{aligned} \tag{A.9}$$

To put this metric into Brinkman form we have to make a change of coordinates as in the previous section,

$$\begin{aligned} \tilde{v} &= \frac{1}{2R^2}\left(x^- + \frac{1}{2}\sum_{i=1}^{4} a_i^{-1}\partial_{x^+}(a_i)(y^i)^2\right), \qquad \hat{\eta}_{1,2} = R\sin\left(\frac{L}{2R^2}\tilde{u}\right)\tilde{\eta}_{1,2} \equiv a_i(u)y_i , \quad \text{(A.10)} \\ \tilde{u} &= 2x^+ , \qquad \hat{\xi}_a = R\cos\left(\frac{L}{R^2}\tilde{u}\right)\tilde{\xi}_a \equiv a_3(u)y_3 , \qquad \hat{x}_1 = R\frac{E}{L}\sin\left(\frac{L}{R^2}\tilde{u}\right)\tilde{x}_1 \equiv a_4(u)y_4 . \end{aligned}$$

and all other coordinates are kept unchanged. So in Brinkman coordinates the metric becomes

$$ds^2 = 2dx^+dx^- - \frac{L^2}{R^4}\left(2\tilde{\psi}_1^2 + 2\tilde{\psi}_2^2 + 4\hat{\xi}_a^2 + \hat{\eta}_1^2 + \hat{\eta}_2^2 + 4\hat{x}_1^2\right)(dx^+)^2 + dE_8^2 . \tag{A.11}$$

Under the above scaling and after the change of coordinates the equations for the embedding of the brane become

$$\begin{aligned} \tilde{\psi}_1 &= \tilde{\psi}_2 = 0 \\ \hat{\eta}_1 &= \hat{\eta}_2 \\ \lambda &= -2R\cot\left(\frac{L}{R^2}x^+\right)\hat{\xi}_b . \end{aligned} \tag{A.12}$$

Hence we see that the limiting brane is oblique, as advertised.

A.3 Various coordinate systems and relations between them

AdS$_{p+2}$: the $p+2$-dimensional AdS space can be viewed as a hyperboloid in $p+3$-dimensional *flat* space with a lorentzian metric of signature $(-,-,+,\ldots,+)$,

$$X_0^2 + X_{p+2} - X_1^2 \cdots - X_{p+1}^2 = R^2 . \tag{A.13}$$

In this paper we use the following two parametrisations of this hyperboloid:

$$\begin{aligned} X_0 &= R\cosh\rho\cos\tau = \frac{1}{2}\left(\frac{1}{u} + u(R^2 + \vec{x}^2 - t^2)\right), \qquad \vec{x} \in R^p , \\ X_{p+2} &= R\cosh\rho\sin\tau = Rut , \\ X_i &= R\sinh\rho\Omega_i = Rux_i , \qquad (i = 1\ldots p) , \\ X_{p+1} &= R\sinh\rho\Omega_{p+1} = \frac{1}{2u}(1 - u^2(R^2 - \vec{x}^2 + t^2)) , \qquad \sum_{i=1}^{p+1}\Omega_i^2 = R^2 . \end{aligned} \tag{A.14}$$

The induced metric on the hyperboloid, written using the second parametrisation, leads to the metric in Poincaré coordinates (2.6), while the first parametrisation leads to the AdS metric in global coordinates

$$ds^2 = R^2\big(- \cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\Omega^2_{S^p}\big)\,. \tag{A.15}$$

For the AdS$_3$ space we have $p = 1$ and we use $\Omega_1 = \cos\sigma$.

AdS$_3$: the AdS$_3$ space is the (universal cover of the) group manifold of the non-compact group SL$(2,\mathbb{R})$. A generic element in this group can be written as

$$g = \frac{1}{R}\begin{pmatrix} X_0 + X_1 & X_2 + X_3 \\ X_2 - X_3 & X_0 - X_1 \end{pmatrix} \tag{A.16}$$

subject to condition that the determinant of (A.16) is equal to one, which is precisely the equation of the hyperboloid given in (A.13).

S^3**:** in the main text we use three different ways to write the metric on S^3. Firstly, using the Euler parametrisation of the group element we have

$$\begin{aligned} g &= e^{i\chi\frac{\sigma_3}{2}} e^{i\tilde\theta\frac{\sigma_1}{2}} e^{i\varphi\frac{\sigma_3}{2}} \\ ds^2 &= \frac{1}{4}[(d\chi + \cos\tilde\theta d\varphi)^2 + d\tilde\theta^2 + \sin^2\tilde\theta d\varphi^2]\,. \end{aligned} \tag{A.17}$$

Secondly, we can use coordinates which are analogue to the global coordinate for AdS$_3$,

$$\begin{aligned} X_1 + iX_2 &= \cos\theta e^{i\tilde\phi}\,, \quad X_3 + iX_4 = \sin\theta e^{i\phi} \\ ds^2 &= d\theta^2 + \cos^2\theta d\tilde\phi^2 + \sin^2\theta d\phi^2\,. \end{aligned} \tag{A.18}$$

The relation between the metrics (A.17) and (A.18) is given by

$$\chi = \tilde\phi + \phi\,, \qquad \varphi = \tilde\phi - \phi\,, \qquad \theta = \frac{\tilde\theta}{2}\,. \tag{A.19}$$

Here the ranges for $\tilde\phi$, ϕ and θ are $-\pi \leq \tilde\phi, \phi \leq \pi$ and $0 \leq \theta \leq \frac{\pi}{2}$ respectively.

Thirdly, the standard metric on S^3 can be written as (using the unit vector $\vec n$ on S^2)

$$\begin{aligned} g &= e^{2i\psi\frac{\vec n\cdot\vec\sigma}{2}}\,, \qquad ds^2 = d\psi^2 + \sin^2\psi(d\xi^2 + \sin^2\xi d\eta^2) \\ X_1 + iX_2 &= \cos\psi + i\sin\psi\cos\xi\,, \qquad X_3 + iX_4 = \sin\psi\sin\xi e^{i\eta}\,. \end{aligned} \tag{A.20}$$

Here the range of coordinates is $0 \leq \psi, \xi \leq \pi$ and $0 \leq \ \eta < 2\pi$.

A.4 General equations of motion for the DBI action

The general Dp brane equations of motion, derived in [20], are given by

$$\partial_i(\sqrt{-\det M}\theta^{ii_1}) = \epsilon^{i_1\cdots i_{p+1}} \sum_{n\geq 0} \frac{1}{n!\,(2!)^n\,(q-1)!}(\mathcal{F})^n_{i_2\cdots i_{2n+1}} \bar F_{i_{2n+2}\cdots i_{p+1}}\,, \tag{A.21}$$

$$\sum_{n\geq 0} \frac{1}{n!\,(2!)^n\,q!} \epsilon^{i_1\cdots i_{p+1}} (\mathcal{F})^n_{i_1\cdots i_{2n}} \bar{F}_{\mu i_{2n+1}\cdots i_{p+1}} =$$

$$= e^{\Phi}\left(\sqrt{-M}(G^{ij}\partial_i X^\nu \partial_j X^\xi g_{\mu\nu}\Phi_{,\xi} - \Phi_{,\mu}) - \mathcal{K}_\mu\right) \qquad \text{(A.22)}$$

where

$$\mathcal{K}_\mu = -\partial_i(\sqrt{-\det M}G^{ij})\partial_j X^\nu g_{\mu\nu} - \sqrt{-\det M}M^{ij}\left(\partial_i\partial_j X^\nu g_{\mu\nu} + \tilde{\Gamma}_{\mu\nu\xi}\partial_i X^\nu \partial_j X^\xi\right), \qquad \text{(A.23)}$$

and $\bar{F}$ is the pull-back of the target space RR fields $C_{[q]}$[16]

$$\bar{F}_{\mu_1\cdots\mu_{q+1}} = (q+1)\partial_{[\mu_1} C_{\mu_2\cdots\mu_{q+1}]} - \frac{(q+1)!}{3!(q-2)!} H_{[\mu_1\mu_2\mu_3} C_{\mu_4\cdots\mu_{(q+1)}]}\,. \qquad \text{(A.24)}$$

The other quantities appearing in the above formulae are given by

$$M_{ij} = \partial_i X^\mu \partial_j X^\nu g_{\mu\nu} + \mathcal{F}_{ij}\,, \qquad \theta^{ij} \equiv M^{[ij]}\,, \qquad G^{ij} \equiv M^{(ij)}\,, \qquad M^{ik}M_{kj} = \delta^i_j\,. \qquad \text{(A.25)}$$

and $\tilde{\Gamma}$ is torsionful connection $\tilde{\Gamma} = \Gamma - \frac{1}{2}H$. Also $\mathcal{F}$ is the gauge invariant two-form

$$\mathcal{F}_{ij} = F_{ij} - \partial_i X^\mu \partial_j X^\nu B_{\mu\nu}\,. \qquad \text{(A.26)}$$

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